Classification of the Dubins set

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Abstract

Given two points in a plane, each with a prescribed direction of motion in it, the question being asked is to find the shortest smooth path of bounded curvature that joins them. The classical 1957 result by Dubins gives a sufficient set of paths (each consisting of circular arcs and straight line segments) which always contains the shortest path. The latter is then found by explicitly computing all paths on the list and then comparing them. This may become a problem in applications where computation time is critical, such as in real-time robot motion planning. Instead, the logical classification scheme considered in this work allows one to extract the shortest path from the Dubins set directly, without explicitly calculating the candidate paths. The approach is demonstrated on one of two possible cases that appear here — when the distance between the two points is relatively large (the case with short distances can be treated similarly). Besides computational savings, this result sheds light on the nature of factors affecting the length of paths in the Dubins problem, and is useful for further extensions, e.g. for finding the shortest path between a point and a manifold in the corresponding configuration space. © 2001 Published by Elsevier Science B.V.

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1. Introduction

Consider the problem of finding the shortest smooth path between two points in the plane, the initial and final points, $P_i$ and $P_f$. Each point is associated with its own orientation angle, $\alpha$ and $\beta$, respectively, which defines the prescribed direction of motion in it (see Fig. 1). The combinations $(P_i, \alpha)$ and $(P_f, \beta)$, called the initial and final configurations, define two points in the corresponding configuration space ($C$-space), and present the problem’s boundary conditions. Given $(P_i, \alpha)$ and $(P_f, \beta)$, the task is to find the shortest smooth path from $P_i$ to $P_f$, such that it starts and ends with the directions of motion $\alpha$ and $\beta$, respectively, and the path curvature is limited by $1/\rho$, where $\rho$ is the minimal radius of turning.

This kind of tasks appear in various applications, such as when joining pieces of railways [1] or planning two- and three-dimensional pipe networks. In robotics, this problem plays a central role in most of the work on nonholonomic motion planning [2–4].

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The complete solution to this problem was first reported in an elegant paper by Dubins [5] in 1957. He showed that any geodesic (i.e. the shortest path) consists of exactly three path segments and presents a sequence $CCC$ or $CSC$, where $C$ (for “circle”) is an arc of radius $\rho$, and $S$ (for “straight”) is a line segment. Each arc $C$ has two options — turning left or turning right. Denote those $L$ and $R$, respectively, and the line segment by $S$. The Dubins set, $D$, includes six admissible paths (or words), $D = \{LSL, RSR, RSL, LSR, RLR, LRL\}$. Furthermore, Dubins’ theorem states that in order to be a candidate for the optimal path, each arc must be of the minimal allowed radius $\rho$.

Using advanced calculus, this result of Dubins was later proved by Reeds and Shepp [6]. Also, Boissonnat et al. [7] proved this result from the standpoint of optimal control, by making use of the powerful Pontryagin’s optimality principle [8]. The more difficult case in which the path from $(P_i, \alpha)$ to $(P_f, \beta)$ can be further shortened by allowing reversals of motion (and thus introducing cusps) was first considered in the same work by Reeds and Shepp [6]. They showed that the initial and final configurations define a sufficient set of 48 paths which contains the optimal path. The technique presented in [9] allows one to pick the optimal solution out of this set of 48 by partitioning the $C$-space into multiple domains such that a single path type is associated with 150 elements and two path types are associated with the other 11 elements.

An alternative approach to the problem with reversals was proposed by Soueres and Laumond [10]. They tie the Pontryagin’s optimality principle with geometric reasoning, and arrive at the optimal solution via partitioning of $C$-space into regions with uniform properties of path optimality.

In the context of robotics, the original Dubins problem of constructing a smooth path has a significance of its own. In many motion planning tasks, such as in the aircraft control, motion reversals are not feasible. Or, if the shortest time path, rather than the shortest path, is desired, the solution is likely to be a smooth path, because the deceleration, stop, and acceleration at the reversal cusps add time to the path execution. Unfortunately, Dubins’ problem with smooth paths is not a subset of the Reeds–Shepp’s problem — the sufficient set of the former is not contained in the sufficient set of the latter. Also, the techniques proposed in [9,10] are not directly applicable to the smooth path case.

To use Dubins’s result for the shortest path calculation, one would need to explicitly calculate the lengths of all arcs and straight line segments in the Dubins set, and then choose the shortest of the computed paths. The time necessary for this calculation may become a bottleneck in time-constrained applications, as e.g. in real-time robot motion planning — which is one motivation for this work. Another motivation comes from problems where one looks for the shortest path from a point to a manifold in the $C$-space. For example, in sensor-based obstacle avoidance, when planning an arrival to some intermediate point $P$ on the obstacle boundary, the current sensing data may suggest that in order not to collide with the obstacle, the orientation angle $\beta$ at $P$ must be within some sector of angles (which may include, e.g. the tangent to the obstacle at $P$). Finding the shortest path to $P$ under this constraint corresponds to finding the shortest path to a line in $C$-space.

In this work, we propose a scheme which allows one to select the shortest path from the Dubins set $D$ directly, without the usual exhaustive calculation of its elements. The scheme is based on a rather suggestive fact, developed in Section 5, that the elements of the Dubins set can be classified into a small number of the so-called equivalency groups, based on the angle quadrants of the corresponding pairs of the initial and final orientation angles. Each equivalency group consists of a few classes of paths, such that any path in a group is equivalent, up to an orthogonal
transformation, to any other path in the same group. This means that the optimal path analysis can be reduced to fewer terms. Further, a simple logical classification of the equivalency groups can be built which points directly to the optimal path.

Below, \( d \) is the Euclidean distance between the initial and final points \( P_i \) and \( P_f \). A rectangular coordinate system \((x, y)\) is chosen such that its origin is \( P_i = (0, 0) \) and the positive direction of \( x\)-axis is toward \( P_f = (d, 0) \) (Fig. 1). The initial and final orientation angles, \( \alpha \) and \( \beta \), are measured counter-clockwise, with respect to the positive direction of \( x\)-axis. Without loss of generality, assume a unit radius of the minimum turning circle, \( \rho = 1 \) (any other \( \rho \) can be reduced to 1 by the scaling \( d = D/\rho \), where \( D \) is the actual distance between \( P_i \) and \( P_f \)). The initial and final arc segments (of radius \( \rho \)) in the Dubins set are denoted \( C_{il}, C_{ir}, C_{fl}, C_{fr} \) (where \( i \) and \( f \) stand for “initial” and “final”, and \( l \) and \( r \) — for “right” and “left”).

The analysis necessary for solving our classification problem turns out to become simpler if it is divided into two cases, which can be called the long path case and the short path case. Our approach is equally applicable to both cases, with minor differences between the resulting computational schemes. For the sake of example, we consider here only one case, the long path case, which seems to be of more interest from the standpoint of applications and the computational savings. More precisely, the “long paths” are those where the distance \( d \) between the points \( P_i \) and \( P_f \) satisfies the condition of non-intersection of the four circles above, \( \{C_{il} \cap C_{ir}\} \cap \{C_{fl} \cap C_{fr}\} = \emptyset \). This covers all cases when \( d > 4 \rho \) and some cases when \( d < 4 \rho \) (see Fig. 4 and Proposition 5).

We first develop, in Section 2, a proper specification scheme for admissible paths. The notion of an equivalency group is then introduced in Section 3, and the scheme for classifying the Dubins set is fully developed in Section 4. This work’s main result which makes this classification possible and becomes the logical scheme for finding the shortest path is summarized in Section 4.

2. Admissible paths and their specification

Given a path from the initial to the final configuration, the position of a point on the path is fully specified by its Cartesian location \( x(\tau), y(\tau) \), where the parameterization variable \( \tau \) can be interpreted as time or the length of path traversed from \( P_i \) with unit velocity. Assume

\[
\tau_i = 0, \text{ i.e. } P_i = P(\tau_i) = P(0);
\]

the point on the path can move only “forward”, from \( P_i \) toward \( P_f \);

it moves with the unit speed;

the orientation angle (direction of motion) cannot change faster than \( 1/\rho \) radian per time unit, where \( \rho = 1 \) is the minimal turning radius.

Following [5], an admissible path is defined as a continuously differentiable curve which is either (i) an arc of a circle of radius 1, followed by a line segment, followed by an arc of a circle of radius 1, or (ii) a sequence of three arcs of circles of radius 1, or (iii) a subpath of a path of type (i) or (ii). A list of admissible paths forms a sufficient set of optimal paths.

To specify admissible paths, we introduce three elementary motions: turning to the left, turning to the right (both along a circle \( C \) of radius 1), and straight line motion \( S \). Also needed will be three corresponding operators, \( L_v \) (for left turn), \( R_v \) (for right turn), \( S_v \) (for straight), which transform an arbitrary point \((x, y, \phi) \in \mathbb{R}^3\) into its corresponding image point in \( \mathbb{R}^3 \),

\[
\begin{align*}
L_v(x, y, \phi) &= (x + \sin(\phi + v) - \sin \phi, y - \cos(\phi + v) + \cos \phi, \phi + v), \\
R_v(x, y, \phi) &= (x - \sin(\phi - v) + \sin \phi, y + \cos(\phi - v) - \cos \phi, \phi - v), \\
S_v(x, y, \phi) &= (x + v \cos \phi, y + v \sin \phi, \phi),
\end{align*}
\]

where index \( v \) indicates that the motion has been along the \((C \text{ or } S)\) segment of length \( v \). With these elementary transformations, any path in the Dubins set \( D = \{LSL, RSR, RSL, LSR, RLR, LRL\} \) can be expressed in terms of
the corresponding equations. In the coordinate system chosen, the initial configuration of each path is at \((0, 0, \alpha)\) and the final configuration at \((d, 0, \beta)\). For example, a path made of segments \(L, R, L\), of the lengths \(t, p, q\), respectively, which starts at point \((0, 0, \alpha)\), must end at \(L_q(R_p(L_t(0, 0, \alpha))) = (d, 0, \beta)\). The length \(\mathcal{L}\) of the path can be defined as the sum of lengths \(t, p\) and \(q\) of its constituent segments,

\[
\mathcal{L} = t + p + q.
\]

Our goal is to classify the elements of set \(\mathcal{D}\) based on the boundary conditions, with the purpose of replacing the explicit computation of all candidates for the shortest path with a simple logical procedure that would directly produce the shortest path. To this end, we will now consider elements of \(\mathcal{D}\) one-by-one and derive the operator equations for the length of each path.

1. \(L_q(S_p(L_t(0, 0, \alpha))) = (d, 0, \beta)\). By applying the corresponding operators (1), this first path in \(\mathcal{D}\) can be represented by a system of three scalar equations:

\[
\begin{align*}
 p \cos(\alpha + t) - \sin \alpha + \sin \beta &= d, \\
 p \sin(\alpha + t) + \cos \alpha - \cos \beta &= 0, \\
 \alpha + t + q &= \beta \mod 2\pi.
\end{align*}
\]

The solution of this system with respect to the segments \(t, p\) and \(q\) is found as

\[
\begin{align*}
 t_{isl} &= -\alpha + \arctan \frac{\cos \beta - \cos \alpha}{d + \sin \alpha - \sin \beta} \mod 2\pi, \\
 p_{isl} &= \sqrt{2 + d^2 - 2 \cos(\alpha - \beta)} + 2d(\sin \alpha - \sin \beta), \\
 q_{isl} &= \beta - \arctan \frac{\cos \beta - \cos \alpha}{d + \sin \alpha - \sin \beta} \mod 2\pi.
\end{align*}
\]

Using definition (2), the length of the path \(LSL\) as a function of the boundary conditions can be now written as

\[
\mathcal{L}_{isl} = t_{isl} + p_{isl} + q_{isl} = -\alpha + \beta + p_{isl}.
\]

2. \(R_q(S_p(R_t(0, 0, \alpha))) = (d, 0, \beta)\). Using (1), we obtain the corresponding scalar equations:

\[
\begin{align*}
 p \cos(\alpha - t) + \sin \alpha - \sin \beta &= d, \\
 p \sin(\alpha - t) - \cos \alpha + \cos \beta &= 0, \\
 \alpha - t + q &= \beta \mod 2\pi.
\end{align*}
\]

The solution of this system, i.e. the lengths of the corresponding segments, is

\[
\begin{align*}
 t_{rsr} &= \alpha - \arctan \frac{\cos \alpha - \cos \beta}{d - \sin \alpha + \sin \beta} \mod 2\pi, \\
 p_{rsr} &= \sqrt{2 + d^2 - 2 \cos(\alpha - \beta)} + 2d(\sin \beta - \sin \alpha), \\
 q_{rsr} &= -\beta \mod 2\pi + \arctan \frac{\cos \alpha - \cos \beta}{d - \sin \alpha + \sin \beta} \mod 2\pi,
\end{align*}
\]

and the path length is given by

\[
\mathcal{L}_{rsr} = t_{rsr} + p_{rsr} + q_{rsr} = \alpha - \beta + p_{rsr}.
\]
3. \( R_q(S_p(L_r(0, 0, \alpha))) = (d, 0, \beta) \). Using (1), we obtain the corresponding scalar equations:

\[
\begin{align*}
    p \cos(\alpha + t) + 2 \sin(\alpha + t) - \sin \alpha - \sin \beta &= d, \\
    p \sin(\alpha + t) - 2 \cos(\alpha + t) + \cos \alpha + \cos \beta &= 0, \\
    \alpha + t - q &= \beta \mod 2\pi.
\end{align*}
\]

The solution of this system is

\[
\begin{align*}
    t_{lsr} &= \left( -\alpha + \arctan \left( \frac{-\cos \alpha - \cos \beta}{d + \sin \alpha + \sin \beta} \right) - \arctan \left( \frac{-2}{p_{lsr}} \right) \right) \mod 2\pi, \\
    p_{lsr} &= \sqrt{-2 + d^2 + 2 \cos(\alpha - \beta) + 2d(\sin \alpha + \sin \beta)}, \\
    q_{lsr} &= -\beta (\mod 2\pi) + \arctan \left( \frac{-\cos \alpha - \cos \beta}{d + \sin \alpha + \sin \beta} \right) - \arctan \left( \frac{-2}{p_{lsr}} \right) \mod 2\pi,
\end{align*}
\]

and the path length is given by

\[
L_{lsr} = t_{lsr} + p_{lsr} + q_{lsr} = \alpha - \beta + 2t_{lsr} + p_{lsr}.
\]

4. \( L_q(S_p(R_t(0, 0, \alpha))) = (d, 0, \beta) \). Using (1), we obtain the corresponding scalar equations:

\[
\begin{align*}
    p \cos(\alpha - t) - 2 \sin(\alpha - t) + \sin \alpha + \sin \beta &= d, \\
    p \sin(\alpha - t) + 2 \cos(\alpha - t) - \cos \alpha + \cos \beta &= 0, \\
    \alpha - t + q &= \beta \mod 2\pi.
\end{align*}
\]

The corresponding solution is

\[
\begin{align*}
    t_{rsl} &= \alpha - \arctan \left( \frac{\cos \alpha + \cos \beta}{d - \sin \alpha - \sin \beta} \right) + \arctan \left( \frac{2}{p_{rsl}} \right) \mod 2\pi, \\
    p_{rsl} &= \sqrt{d^2 - 2 + 2 \cos(\alpha - \beta) - 2d(\sin \alpha + \sin \beta)}, \\
    q_{rsl} &= \beta (\mod 2\pi) - \arctan \left( \frac{\cos \alpha + \cos \beta}{d - \sin \alpha - \sin \beta} \right) + \arctan \left( \frac{2}{p_{rsl}} \right) \mod 2\pi,
\end{align*}
\]

and the path length is given by

\[
L_{rsl} = t_{rsl} + p_{rsl} + q_{rsl} = -\alpha + \beta + 2t_{rsl} + p_{rsl}.
\]

5. \( R_q(L_p(R_t(0, 0, \alpha))) = (d, 0, \beta) \). Using (1), we obtain the corresponding scalar equations:

\[
\begin{align*}
    2 \sin(\alpha - t + p) - 2 \sin(\alpha - t) &= d - \sin \alpha + \sin \beta, \\
    -2 \cos(\alpha - t + p) + 2 \cos(\alpha - t) &= \cos \alpha - \cos \beta, \\
    \alpha - t + p - q &= \beta \mod 2\pi.
\end{align*}
\]
The solution of this system is
\[ t_{rlr} = \alpha - \arctan \left( \frac{\cos \alpha - \cos \beta}{d - \sin \alpha + \sin \beta} \right) + \frac{p_{rlr}}{2} \text{[mod } 2\pi], \]
\[ p_{rlr} = \arccos \frac{1}{2}(6 - d^2 + 2\cos(\alpha - \beta) + 2d(\sin \alpha - \sin \beta)), \]
\[ q_{rlr} = \alpha - \beta - t_{rlr} + p_{rlr} \text{[mod } 2\pi], \]
and the path length is obtained by substituting (11) into (2),
\[ L_{rlr} = t_{rlr} + p_{rlr} + q_{rlr} = \alpha - \beta + 2p_{rlr}. \] (12)

6. \( L_q(R_p(L_t(0, 0, \alpha))) = (d, 0, \beta). \) Using (1), we obtain the corresponding scalar equations:
\[-2\sin(\alpha + t - p) + 2\sin(\alpha + t) = d + \sin \alpha - \sin \beta, \]
\[2\cos(\alpha + t - p) - 2\cos(\alpha + t) = -\cos \alpha + \cos \beta, \]
\[\alpha + t - p + q = \beta \text{[mod } 2\pi].\]
The corresponding solution is
\[ t_{btl} = \left(-\alpha + \arctan \left( \frac{-\cos \alpha + \cos \alpha}{d + \sin \alpha - \sin \beta} \right) + \frac{p_{btl}}{2} \right) \text{[mod } 2\pi], \]
\[ p_{btl} = \arccos \frac{1}{2}(6 - d^2 + 2\cos(\alpha - \beta) + 2d(\sin \alpha - \sin \beta)) \text{[mod } 2\pi], \]
\[ q_{btl} = \beta \text{[mod } 2\pi] - \alpha + 2p_{btl} \text{[mod } 2\pi], \] (13)
and the path length is given by
\[ L_{btl} = t_{btl} + p_{btl} + q_{btl} = -\alpha + \beta + 2p_{btl}. \] (14)

3. Equivalency groups

We are now prepared to turn to the classification of the Dubins set \( D. \) Divide the range of possible orientation angles \((\alpha, \beta)\) into four quadrants; Fig. 1: quadrant 1 corresponds to the range \([0, \pi/2]\), quadrant 2 to \([\pi/2, \pi]\), quadrant 3 to \([\pi, 3\pi/2]\), and quadrant 4 to the range \([3\pi/2, 2\pi]\). Since each of \(\alpha\) or \(\beta\) can be in any of the four quadrants, together this produces 16 different combinations of possible quadrants. We represent those 16 by a \(4 \times 4\) matrix, \(a_{ij}\), where index \(i\) corresponds to the quadrant number of the initial, and index \(j\) that of the final orientation. Element \(a_{ij}\) therefore describes the class of all paths whose initial and final orientation angles \((\alpha, \beta)\) belong to the quadrants \(i\) and \(j\), respectively. For example, the case \(\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]\) corresponds to the element \(a_{12}\) and covers all those paths whose orientation angles belong to the first and second quadrants, respectively.

It will be shown below that these 16 classes can be reduced to six independent clusters, called equivalency groups, such that an orthogonal transformation of any path in a given group changes it into a path in the same or a different class of the same group.

Dubins’ main theorem [5] says that each (non-degenerate) candidate for the optimal path in set \( D \) must start with a piece of circle and end with a piece of circle (of radius \(\rho = 1\), see above). Depending on the path in \( D \), the initial and the final circle can turn either left or right; we denote those \(C_{il}, C_{ir}\) and \(C_{fl}, C_{fr}\), respectively. To proceed, we will need the following definition.

**Definition.** Two paths are topologically equivalent (denoted by “\(\simeq\)”) if there exists an orthogonal transformation that maps one path into the other, with both paths sharing their respective initial and final points.
Fig. 2. Arcs \( P_1 P_2 \) and \( P_3 P_f \) are obtained by orthogonal transformations of the arc \( P_1 P_2 \); the first is a mirror reflection of \( P_1 P_2 \) with respect to line \( (P_2 P_f) \), the second — the central symmetry reflection of \( P_1 P_2 \) with respect to point \( O \). Arc \( P_4 P_f \) is obtained by applying a composition of both transformations.

Note that two equivalent paths are of the same length and allow equivalent parameterization. The following proposition relates the topological equivalency of paths and their initial/final configurations.

**Proposition 1.** For any path connecting two points, \( P(t_i, \alpha), P(t_f, \beta) \), where \((\alpha, \beta)\) are the initial and final orientation angles, there exist another three paths which are topologically equivalent to it. Their corresponding orientation angles are \((-\alpha, -\beta), (\beta, \alpha), \) and \((\beta, \alpha)\).

To see this, consider a path that starts at an initial configuration \((P_i, \alpha)\) and is of the form \( P_i P_1 \cdots [-\alpha, -\beta] \); here the ellipsis “…” reflect our emphasis on the segment \( P_i P_1 \) (Fig. 2). By applying a mirror reflection \( G_{(P_i P_f)} \) with respect to the line \( (P_i P_f) \), this path is transformed into the path \( P_i P_2 \cdots [-\alpha, -\beta] \). Similarly, by applying the central symmetry reflection \( G_{(O)} \) with respect to the midpoint \( O \) of segment \([P_i, P_f]\), the same path is transformed into \( \cdots P_4 P_f [\beta, \alpha] \). The composition of both transformations leads to
\[
G_{(O)}(G_{(P_i P_f)}(P_1 P_i \cdots [\alpha, \beta])) = G_{(P_i P_f)}(G_{(O)}(P_1 P_i \cdots [\alpha, \beta])) = \cdots P_4 P_f [-\beta, -\alpha].
\]

This general fact will be used below in the analysis of paths defined by set \( D \). Recall that those paths take a form either \( CCC \) or \( CSC \), where \( C \) is an arc of a circle of radius \( \rho \) with options \( L \) and \( R \) (left and right), and \( S \) is a straight line segment. To distinguish between the first and the second arc segments in the path \( CSC \), subscripts will be used, \( C_1 SC_2 \).

Define the *conjugate* of \( C_1 \), denoted \( \tilde{C}_1 \), as the complement of \( C \), i.e. if \( C_1 = R \) then its conjugate is \( \tilde{C}_1 = L \), and vice versa. The application of the orthogonal transformations \( G_{(P_i P_f)} \) and \( G_{(O)} \) leads to
\[
G_{(P_i P_f)}(C_1 SC_2 [\alpha, \beta]) = \tilde{C}_1 \tilde{S} \tilde{C}_2 [-\alpha, -\beta], \quad G_{(O)}(C_1 SC_2 [\alpha, \beta]) = \tilde{C}_2 \tilde{S} \tilde{C}_1 [\beta, \alpha].
\]

Notice that the mirror reflection \( G_{(P_i P_f)} \) reverses the signs of angles \( \alpha, \beta \) and changes the arc segments to their conjugates. The central symmetry reflection \( G_{(O)} \) has a triple effect: it switches orientations \( \alpha \) and \( \beta \), switches segments \( C_1 \) and \( C_2 \), and switches each segment to its conjugate. This relation can be proven rigorously by formalizing the operators \( G_{(P_i P_f)} \) and \( G_{(O)} \) and then applying them to the path presented in the general operator form, as e.g. \( L_q(S_p(L_2(0, 0, \alpha))) = (d, 0, \beta) \).

Independent of the order of transformations, the composition of \( G_{(P_i P_f)} \) and \( G_{(O)} \) leads to
\[
G_{(O)}(G_{(P_i P_f)}(C_1 SC_2 [\alpha, \beta])) = G_{(P_i P_f)}(G_{(O)}(C_1 SC_2 [\alpha, \beta])) = C_2 SC_1 [-\beta, -\alpha].
\]

Using the above definition of topological equivalence, the following proposition defines the set of topologically equivalent paths.
Proposition 2. Given the path \( C_1 S C_1[\alpha, \beta] \), the composition of orthogonal transformations \( G_{(P, P_f)} \) and \( G_{(O)} \) leads to the topologically equivalent paths

\[ C_1 S C_2[\alpha, \beta] \simeq \tilde{C}_1 \tilde{S} \tilde{C}_2[-\alpha, -\beta] \simeq \tilde{C}_2 \tilde{S} \tilde{C}_1[\beta, \alpha] \simeq C_2 S C_1[-\beta, -\alpha], \]

where \( \alpha, \beta \) are the initial and final orientations, \( G_{(P, P_f)} \) the mirror reflection with respect to line \( (P, P_f) \), \( G_{(O)} \) the central symmetry reflection with respect to the midpoint \( O \) of segment \( [P_1, P_f] \), and “\( \simeq \)” the sign of equivalency.

Example. Consider the paths shown in Fig. 3. Initially there are two paths, \( RSR[\alpha, \beta] \) and \( RSL[\alpha, \beta] \) (Fig. 3(a)), where \( \alpha \in [0, \pi/2] \), \( \beta \in [\pi/2, \pi] \).

1. Consider first the path \( RSR[\alpha, \beta] \). With the notation of Proposition 2, we have \( C_1 = R \) and \( C_2 = R \). The proposition gives the following set of topologically equivalent paths (see Fig. 3(a)–(d)):

\[ RSR[\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]] \simeq LSL[\alpha \in [3\pi/2, 2\pi], \beta \in [\pi, 3\pi/2]] \simeq LSL[\alpha \in [\pi/2, \pi], \beta \in [0, \pi/2]] \simeq RSR[\alpha \in [\pi, 3\pi/2], \beta \in [3\pi/2, 2\pi]]. \]

2. Consider now an example \( C_1 = R \), \( C_2 = L \). By applying Proposition 2 to the path \( RSL[\alpha, \beta] \) with \( \alpha \in [0, \pi/2] \), \( \beta \in [\pi/2, \pi] \) (Fig. 3(a)), we obtain three other paths (Fig. 3(b)–(d)):...
Theorem 1

It is convenient to combine Propositions 2 and 3 into one theorem. That is, the central symmetry reflection, $C_2$. Show that the condition 1, when satisfied, leads to a further simplification of the set of candidates for the optimal composition of $G$.

Composition lead to the topologically equivalent paths $C_1SC_2[\alpha, \beta]$ to class $c_1$. Proposition 4. Matrix $a_{ij}$ can be divided into six independent equivalency groups: (1) $a_{11} \simeq a_{44}$, (2) $a_{12} \simeq a_{21} \simeq a_{34} \simeq a_{43}$, (3) $a_{13} \simeq a_{24} \simeq a_{31} \simeq a_{42}$, (4) $a_{14} \simeq a_{41}$, (5) $a_{22} \simeq a_{33}$, and (6) $a_{23} \simeq a_{32}$.

Indeed, according to the Transformation Theorem, any path with $\alpha \in [0, \pi/2]$, $\beta \in [0, \pi/2, \pi]$ (i.e. belonging to class $a_{11}$) is transformed into an equivalent path with $\alpha \in [3\pi/2, 2\pi]$, $\beta \in [3\pi/2, 2\pi]$ (which is from class $a_{44}$). That is, the central symmetry reflection, $G(O)$, leads to a topologically equivalent path from class $a_{11}$, while the composition of $G(P, P_f)$ and $G(O)$ leads to a topologically equivalent path from class $a_{44}$. In the case of the equivalency group (2), for any path of class $a_{12}$ there exists an equivalent path in each of the classes $a_{21}$, $a_{34}$, and $a_{43}$.

By choosing one representative from each equivalency group, we define a basis set $B$ of matrix $a_{ij}$ — a list of six mutually independent classes of orientation pairs $\alpha$, $\beta$. This reduces from 16 to 6, the number of path classes to be analyzed for the optimal solution. Note that the basis set is not unique since its members can be chosen in various ways.

4. Classes of paths and their equivalency groups

The above scheme for classifying the Dubins set will be fully developed in this section; the necessary analysis involves the following steps:

1. Find the necessary and sufficient condition of non-intersection of the unions $C_{\alpha} \cup C_{\alpha'}$ and $C_{\beta} \cup C_{\beta'}$. This condition formally defines what is meant by the “long paths” in the case under study.

2. Show that the condition 1, when satisfied, leads to a further simplification of the set of candidates for the optimal solution that need be considered.
Proposition 5. Follows.

Since elements from the same equivalence group have similar properties, the development steps 3 and 4 are combined below for each of the groups, forming six corresponding subsections.

When applying results of this analysis to a specific problem, one would proceed as follows:

- Make sure that the task at hand satisfies the condition in step 1.
- Associate the given initial and final orientations with a class \( a_{ij} \). For class \( a_{ij} \), use the uniquely defined optimal path.

Below, after analyzing steps 1 and 2, for the sake of convenience we choose for step 3, a particular example of basis set \( B \), \( B = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{22}, a_{23}\} \). For each of the six elements of \( B \), questions posed in steps 3 and 4 are then addressed in the respective six sections.

The following additional notation is used below: unless stated otherwise, forms like \( AB \) and \( A'B' \) represent straight line segments and circular arc segments, respectively, with \( A \) and \( B \) being the segments’ endpoints. When in mathematical expressions, the same forms denote the lengths of the segments. When needed for clarity, the strings may be longer: e.g. \( A_1 B_1 C_1 D_1 \) is an arc with the endpoints \( A_1, D_1 \) and two inner points \( B_1, C_1 \).

4.1. The long path case

Turning to the step 1 above, the condition of non-intersection of union \( \{C_{il} \cup C_{ir}\} \) with union \( \{C_{fl} \cup C_{fr}\} \) is as follows.

**Proposition 5.** \( \{C_{il} \cup C_{ir}\} \cap \{C_{fl} \cup C_{fr}\} = \emptyset \) if \( d > \sqrt{4 - (|\cos \alpha| + |\cos \beta|)^2} + |\sin \alpha| + |\sin \beta| \). This condition on \( d \) is a precise definition of the long path case.

To prove this necessary and sufficient condition, consider the case when the union \( \{C_{il} \cup C_{ir}\} \) is tangent to the union \( \{C_{fl} \cup C_{fr}\} \), i.e. there exist only one point belonging to both unions. Take, for instance, the case \( a_{11} \) (\( \alpha \) and \( \beta \) are in the first quadrant, Fig. 4). Assume a unit radius, \( \rho = 1 \). Given a common tangent to both arcs, \( IF = IA + AB + BF \). From \( \triangle IAO_1 \); \( IA = \sin \alpha \) and \( O_1A = \cos \alpha \). From \( \triangle FBO_2 \); \( BF = \sin \beta \) and \( O_2B = \cos \beta \). From \( \triangle O_1O_2C \); \( O_1O_2 = 2, O_2C = O_2B + O_1A \), and therefore \( O_1C = \sqrt{4 - (\cos \alpha + \cos \beta)^2} \). Summing up for \( IA + AB + BF \), obtain the expression for distance \( IF \), which is a condition for a common tangent for the right initial circle and left final circle. In general, the expression \( \sqrt{4 - (|\cos \alpha| + |\cos \beta|)^2} + |\sin \alpha| + |\sin \beta| = d \) covers all possible cases of paths consisting of circular arcs with a common tangent point.

**Proposition 6.** For the long path case, the path CCC cannot be the optimal solution.

To show this, consider the basis set \( B \) of independent orientation pairs \( (\alpha, \beta) \). We need to show that for any element from \( B \), there exists a path of type \( CSC \) that is shorter than the path CCC.

Since for the long path case, \( \{C_{il} \cup C_{ir}\} \cap \{C_{fl} \cup C_{fr}\} = \emptyset \), then \( C_{il}, C_{ir}, C_{fl}, C_{fr} \) do not intersect. Fig. 5 illustrates this for the general case: it is clear in this example that though the path CCC is physically realizable, it can be excluded from the list of candidates for an optimal solution.

To prove this, assume that circle \( C_{left} \) is one of the initial circles \( C_{il} \) or \( C_{ir} \), and \( C_{right} \) is either \( C_{fl} \) or \( C_{fr} \). Two other circles tangent to \( C_{left} \) and \( C_{right} \) are \( C_{up} \) (upper tangent) and \( C_{down} \) (lower tangent). The initial and final orientations can be chosen either clockwise or counter-clockwise. Notice that the orientation of the initial and final circles has to be the same, since the path of type CCC switches directions when passing from one circle to another. If two switchings take place, the initial and final circles must have the same orientation.
The area of possible initial and final positions is limited; the first must lie on the arc $A_1B_1C_1D_1$ of circle $C_{\text{left}}$, and the second on the segment $A_2B_2C_2D_2$ of circle $C_{\text{right}}$ (both segments are shown in solid line in Fig. 5). These restrictions are dictated by Proposition 5.

Let us say, the initial and final positions lie within the arcs $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, respectively, both with the same counter-clockwise orientations. (For the clockwise case, the analysis is similar.) There are four options: (i) the initial position $P_{ti}$ belongs to segment $A_1B_1C_1$ and the final position $P_{tf}$ to segment $A_1B_2C_2$, (ii) $P_{ti} \in C_1D_1$ and $P_{tf} \in C_2D_2$, (iii) $P_{ti} \in C_1D_1$ and $P_{tf} \in A_2B_2C_2$, and (iv) $P_{ti} \in A_1B_1C_1$ and $P_{tf} \in C_2D_2$.

In case (i) the solution $LSL$ is the shortest possible path, since the straight line segment connecting $C_1$ and $C_2$ is shorter than the sum of segments $C_1D_1 + D_1D_2 + D_2C_2$. In case (ii), the path $LRL$ cannot be the optimal solution: since the middle arc is less than $\pi/2$, this path is of type $CCC$, which was shown above to disqualify it from being a...
candidate for the optimal solution. For the same reason, path CCC cannot be the optimal solution in cases (iii) and (iv).

4.2. The equivalency groups of \( \{a_{ij}\} \) classes

We now turn to defining the equivalency groups for each element of the matrix \( \{a_{ij}\} \). Recall that each element represents a class of paths.

4.2.1. Equivalency group \( a_{11}, a_{44} \)

According to Proposition 4, classes \( a_{11} \) and \( a_{44} \) belong to the same equivalency group. We first show for class \( a_{11} \) that the corresponding Dubins set can be reduced directly to a unique optimal solution. Then, by applying an orthogonal transformation to the optimal path for class \( a_{11} \), the optimal solution for \( a_{44} \) will be obtained.

**Proposition 7.** For the long path case (see Proposition 5), the optimal solution corresponding to the element \( a_{11} \) is RSL.

Note that the number of candidate curves for the optimal path in set \( \mathcal{D} \) is now reduced to LSL, LSR, RSL, RSR — the curves of type CCC are excluded from consideration (Proposition 6). For three of those paths, LSL, LSR, RSR, the \( x \)-coordinate goes outside the range \( 0 \leq x \leq d \). Take the length of the curve LS as the lower bound on the length of the paths LSL, LSR and RSR (LS is a subpath of paths LSL or LSR with \( \alpha = \pi/2 \)).

It is claimed that the upper bound on the length of paths for which the \( x \)-coordinate is in the range \( 0 \leq x \leq d \) is the path RSL with \( \alpha = \beta = \pi/2 \). Indeed, from Section 2, \( \partial L_{rsl}/\partial \alpha > 0 \) and \( \partial L_{rsl}/\partial \beta > 0 \). Therefore, the maximum of \( L_{rsl} \) in this region will occur when \( \alpha \) and \( \beta \) are equal to \( \pi/2 \).

To prove that the optimal solution for paths within the range \( 0 \leq x \leq d \) is RSL, one needs to show that the lower bound on the path length in the Dubins subset \( \{LSL, LSR, RSR\} \) is bigger than the upper bound on the path RSL. This case is illustrated in Fig. 6.

**Lemma 1.** If \( \alpha \) and \( \beta \) are in the first quadrant then the upper bound for \( L_{rsl} \) is limited by

\[
\max_{\alpha, \beta \in [0, \pi/2]} L_{rsl} \leq \sqrt{d^2 - 4d} + 2\pi.
\]

Indeed, when \( \alpha \in [0, \pi/2] \) and \( \beta \in [0, \pi/2] \), the gradient of the path RSL is a positive function. This means that the function is monotonically increasing on that interval and reaches its maximum on the interval’s boundary, i.e. \( \alpha = \beta = \pi/2 \). Assuming, as usual, \( \rho = 1 \), the upper bound for the path length is

\[
L_{rsl} = IA + AB + B’F,
\]

where

\[
IA + B’F < 2\pi.
\]
Segment $AB$ can be found as $AB = 2AD$ or, expressing $AD$ in terms of $O_1D$ and $O_1A$,
\[
AB = 2\sqrt{\left(\frac{1}{2}(d - 2)\right)^2 - 1} = \sqrt{d^2 - 4d}.
\]  
(18)

Substituting (17) and (18) into (16), we obtain
\[
\max_{\alpha, \beta \in [0, \pi/2]} \mathcal{L}_{rl} \leq \sqrt{d^2 - 4d} + 2\pi.
\]

**Lemma 2.** If $\alpha$ and $\beta$ are in the first quadrant then the lower bound on $\mathcal{L}_{lsl}$, $\mathcal{L}_{lsr}$, $\mathcal{L}_{rsr}$ is
\[
\min_{\alpha, \beta \in [0, \pi/2]} \{\mathcal{L}_{lsl}, \mathcal{L}_{lsr}, \mathcal{L}_{rsr}\} \geq \sqrt{d^2 + 2d + 3\pi/2}.
\]

The lower bound on $LSL$, $LSR$ and $RSR$ can be obtained by taking $\alpha = \pi/2$ and $\beta$ as shown in Fig. 6. The minimum path length can then be estimated as
\[
\min_{\alpha, \beta \in [0, \pi/2]} \{\mathcal{L}_{lsl}, \mathcal{L}_{lsr}, \mathcal{L}_{rsr}\} = I\bar{EC} + CF > 3\pi/2 + \sqrt{d^2 + 4d + 1} - 1 > 3\pi/2 + \sqrt{d^2 + 2d}.
\]  
(19)

In order to prove that the optimal solution corresponding to class $a_{11}$ is $RSR$, we need to show that
\[
\min_{\alpha, \beta} \{\mathcal{L}_{lsl}, \mathcal{L}_{lsr}, \mathcal{L}_{rsr}\} - \max_{\alpha, \beta} \mathcal{L}_{rl} > 0.
\]  
(20)

It is easy to see that (20) holds if $\sqrt{d^2 + 2d} - \sqrt{d^2 - 4d} - \pi/2 > 0$: move $\pi/2$ to the right side of the inequality and multiply both sides by the positive expression $\sqrt{d^2 + 2d} + \sqrt{d^2 - 4d}$. The result is $6d > \pi \sqrt{d^2 + 2d}$, which is true if $d^2 - 4d > 0$ — precisely the case we are interested in. This completes the proof of inequality (20) and of the claim that the optimal solution for class $a_{11}$ is $RSR$. By applying Proposition 2 to the path $RSR$, obtain a similar statement for class $a_{44}$.

**Proposition 8.** Given that $a_{11}$ and $a_{44}$ are in the same equivalency group and the optimal solution for $a_{11}$ is $RSR$, the optimal solution for $a_{44}$ is $LSL$.

4.2.2. Equivalency group $\{a_{12}, a_{21}, a_{34}, a_{43}\}$

According to Proposition 4, path classes $a_{12} \simeq a_{21} \simeq a_{34} \simeq a_{43}$ are in the same equivalency group. We first show how to extract the optimal solution for class $a_{12}$; it turns out that class $a_{12}$ defines two (rather than one as with class $a_{11}$) elements of the Dubins set as candidates for the optimal solution. Accordingly, a switching function $S_{12}$ will derive whose sign will uniquely determine which of the two is the optimal solution. Then, by applying the orthogonal transformation to the paths of class $a_{12}$ (see Proposition 2), optimal solutions for path classes $a_{21}$, $a_{34}$, and $a_{43}$ will be obtained.

**Proposition 9.** For the long path case, the optimal solution corresponding to the class $a_{12}$ is either $RSR$ or $LSR$.

It follows from Proposition 6 that the path of type $CCC$ can be excluded from consideration. This leaves four candidates, $RSR$, $RSR$, $LSR$, and $LSL$. Define the critical initial orientation as one where orientation $\alpha$ coincides with the tangent to the circle $O_{RF}$; denote it $\alpha = \bar{\alpha}$. Note that the set $(\beta, d)$ uniquely defines the critical initial orientation, $\bar{\alpha} = \bar{\alpha}(\beta, d)$. If $\alpha > \bar{\alpha}$ then the path $LSR$ is not feasible and can be excluded from consideration; otherwise, path $RSR$ should be excluded. Consider the case when path $LSR$ is feasible ($\alpha < \bar{\alpha}$). If $\alpha = 0$ and $\beta = \pi$ then the length of $LSR$ is equal to that of $RSR$. Analysis of gradients of functions $\mathcal{L}_{rsr}$ and $\mathcal{L}_{lsl}$ shows that if $\alpha$ is increasing or $\beta$ is decreasing then path $RSR$ becomes shorter than $LSR$. This is true until $\alpha$ reaches the critical initial orientation $\alpha = \bar{\alpha}$. Comparing the lower bound on the length of path $LSL$ and the upper bound on the length of path...
To find now the optimal solution for class $a_{12}$, we define a switching function, $S_{12}$. The following proposition holds.

**Proposition 10.** For class $a_{12}$ (i.e. $0 < \alpha \leq \pi / 2$, $\pi / 2 < \beta \leq \pi$), the optimal solution is RSR if $S_{12}(p_{rsr}, p_{rsl}, q_{rsl}) < 0$, and it is RSL if $S_{12}(p_{rsr}, p_{rsl}, q_{rsl}) > 0$, with

$$S_{12}(p_{rsr}, p_{rsl}, q_{rsl}) = p_{rsr} - p_{rsl} - 2(q_{rsl} - \pi),$$

where $p_{rsr}, p_{rsl},$ and $q_{rsl}$ are defined by (5) and (9).
Remark. For class \( a_{12} \), the region of \( \alpha, \beta \) where \( RSR \) is the optimal solution is much smaller than the corresponding region for \( RSL \). To save on computations, divide the whole region into two subregions: one where \( RSL \) is the optimal solution, and the other (much smaller), where the solution is \( RSL \) or \( RSR \) and depends on the condition of Proposition 9. The first subregion (where \( RSL \) is the optimal solution) is defined by

\[
d \cos \beta - 3 \sin(\beta) \cos \beta + \sin(\beta - \alpha) + \cos \alpha \sin \beta > 0.
\]

This case occurs if \( \alpha \) and \( \beta \) are such that the last arc in the path \( RSL \), \( q_{rsl} \), is equal to \( \pi \). If \( q_{rsl} < \pi \) then path \( RSL \) is shorter than path \( RSR \). If \( q_{rsl} > \pi \), then the switching function \( S_{12} \) needs be checked.

We now turn to the classes \( a_{21}, a_{34} \), and \( a_{43} \) which are in the same equivalency group as \( a_{12} \). By applying the orthogonal transformation (see Proposition 2), the set of path candidates for \( a_{12} \) is transformed into the set of candidates for the remaining elements of the equivalency group \( a_{21}, a_{34}, a_{43} \) (see Fig. 7). That leads to the following result.

**Proposition 11.** Since \( a_{12} \simeq a_{21} \simeq a_{34} \simeq a_{43} \) and the optimal solution for \( a_{12} \) is \{\( RSL \) or \( RSR \)\}, then the optimal solutions for the remaining elements of this equivalency group are \( a_{21} \mapsto \{RSL\text{ or }RSR\}; a_{34} \mapsto \{LSR\text{ or }RSL\}; a_{43} \mapsto \{LSR\text{ or }LSL\} \).

The respective switching functions for classes \( a_{21}, a_{34} \), and \( a_{43} \) (see the next three propositions) are obtained from the switching function for class \( a_{12} \), by replacing \( t \) and \( q \) segments as prescribed by the transformation theorem.

**Proposition 12.** For class \( a_{21} \) (i.e. \( \pi/2 < \alpha \leq \pi \), \( 0 < \beta \leq \pi/2 \)), the optimal solution is \( LSL \) if \( S_{21} < 0 \), and it is \( RSL \) if \( S_{21} > 0 \), with

\[
S_{21}(p_{isl}, p_{rsl}, t_{rsl}) = p_{isl} - p_{rsl} - 2(t_{rsl} - \pi). \tag{22}
\]

To check, apply Theorem 1 to the switching function \( S_{12} = L_{rsr} - L_{rsl} = p_{rsr} - p_{rsl} - 2\gamma \), where \( \gamma = q_{rsl} - \pi \); obtain \( L_{rsl} = SA + p_{rsl} + \xi + \pi \) and \( L_{rsr} = SA + \xi + \gamma + p_{rsr} + \gamma + \pi \). According to Proposition 3, the orthogonal transformation implies

\[
L_{rsl}(\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]) = L_{isl}(\alpha \in [\pi/2, \pi], \beta \in [0, \pi/2]),
\]

and, additionally, \( p_{rsr}(\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]) = p_{isl}(\alpha \in [\pi/2, \pi], \beta \in [0, \pi/2]) \) and \( p_{rl}(\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]) = p_{rsl}(\alpha \in [\pi/2, \pi], \beta \in [0, \pi/2]), \) and \( q_{rsr}(\alpha \in [0, \pi/2], \beta \in [\pi/2, \pi]) = t_{rsr}(\alpha \in [\pi/2, \pi], \beta \in [0, \pi/2]) \). The switching function for class \( a_{21} \) can be simply obtained from the function \( S_{12} \) by changing the segments from \( p_{rsr}, p_{rsl}, q_{rsl} \) to \( p_{isl}, p_{rsl}, t_{rsl} \), respectively. The validity of this procedure follows from the transformation theorem, and using again the same theorem, obtain directly the switching functions for classes \( a_{34} \) and \( a_{43} \).

**Proposition 13.** For class \( a_{34} \) (i.e. \( \pi < \alpha \leq 3\pi/2, 3\pi/2 < \beta \leq 2\pi \)), the optimal solution is \( RSR \) if \( S_{34} < 0 \), and it is \( LSR \) if \( S_{34} > 0 \), with

\[
S_{34}(p_{rsr}, p_{isl}, t_{isl}) = p_{rsr} - p_{isl} - 2(t_{isl} - \pi). \tag{23}
\]

**Proposition 14.** For class \( a_{43} \) (i.e. \( 3\pi/2 < \alpha \leq 2\pi, \pi < \beta \leq 3\pi/2 \)), the optimal solution is \( LSL \) if \( S_{43} < 0 \), and it is \( LSR \) if \( S_{43} > 0 \), with

\[
S_{43}(p_{isl}, p_{lsl}, q_{lsl}) = p_{isl} - p_{lsl} - 2(q_{lsl} - \pi). \tag{24}
\]
4.2.3. Equivalency group \( \{a_{13}, a_{24}, a_{31}, a_{42}\} \)

According to Proposition 4, classes \( a_{13}, a_{24}, a_{31}, a_{42} \) belong to the same equivalency group. First we show that for class \( a_{13} \) the Dubins set can be reduced to two path candidates, and then further reduced to a unique optimal solution using an appropriate switching function, \( S_{13} \). Then, by applying the transformation theorem, the switching function \( S_{13} \) will be modified to produce the corresponding switching functions for classes \( a_{24}, a_{31}, a_{42} \). For \( a_{13} \), the following holds.

**Proposition 15.** For the long path case, the optimal solution corresponding to the element \( a_{13} \) is either RSR or LSR.

Similar to the argument for class \( a_{12} \) above, one can see that path RSL can be excluded from the set of candidates considered for the optimal path. Indeed, if \( \alpha = 0 \) and \( \beta = \pi \), then \( L_{lsl} = L_{rsl} \). If \( \alpha \) or \( \beta \) are increasing then path LSR becomes shorter than path RSL. To see this, compare the line segment \( SD \), which is a tangent to circle \( O_{LF} \), with \( SD \), a tangent to circle \( O_{RF} \) (see Fig. 8(a)). Now, \( SA < SD \) for any \( \beta \in (\pi, 3\pi/2] \), and \( A'B < D'C \); also, \( SD < S'G + GE + E'D \). This leads to 

\[
SA + A'B < S'G + GE + E'C.
\]

Therefore, if \( \alpha \in [0, \tilde{\alpha}] \) then \( L_{ISR} < L_{rsl} \). Similarly, if \( \alpha > \tilde{\alpha} \) then \( L_{rsl} < L_{rsl} \). Here \( \tilde{\alpha} \) is the critical angle \( \alpha = \tilde{\alpha}(d, \beta) \), where paths LSR and RSR degenerate to SR. This implies that only paths LSR and RSR are candidates for the optimal solution.

Using Proposition 2, a similar argument extends this result to the remaining three elements, as follows.
Proposition 16. Since \( a_{13} \simeq a_{24} \simeq a_{31} \simeq a_{42} \) and the optimal solution for \( a_{13} \) is either RSR or LSR, the optimal solutions for \( a_{24} \), \( a_{31} \), and \( a_{42} \) are, respectively,

\[
a_{24} \mapsto \{ \text{RSR or RSL} \}; \ a_{31} \mapsto \{ \text{LSL or LSR} \}; \ a_{42} \mapsto \{ \text{LSL or RSL} \}.
\]

Turning back to class \( a_{13} \), we can verify that the critical angle \( N \) defines switching of the optimal path from LSR to RSR; i.e. if \( \alpha > \bar{\alpha} \) the optimal solution is RSR, otherwise it is LSR. Another indicator of the switch from LSR to RSR is the length of arc \( t_{\text{rsr}} \). Notice that if RSR is the optimal solution for class \( a_{13} \) then \( t_{\text{rsr}} \) cannot exceed \( \pi \).

This observation leads to the corresponding switching functions necessary for obtaining the optimal solution.

Proposition 17. For class \( a_{13} \) (i.e. \( 0 < \alpha \leq \pi/2 \), \( 3\pi/2 < \beta \leq 2\pi \)), if

1. \( S_{13}(t_{\text{rsr}}) < 0 \), then the optimal solution is RSR,
2. \( S_{13}(t_{\text{rsr}}) > 0 \), then the optimal solution is LSR,

where

\[
S_{13}(t_{\text{rsr}}) = t_{\text{rsr}} - \pi.
\]

The transformation theorem extends the results obtained for class \( a_{13} \) to the accompanying classes \( a_{24} \), \( a_{31} \) and \( a_{42} \).

Proposition 18. For class \( a_{24} \) (i.e. \( \pi/2 < \alpha \leq \pi \), \( 3\pi/2 < \beta \leq 2\pi \)), the optimal solution is RSR if \( S_{24} < 0 \), and it is RSL otherwise, where

\[
S_{24}(q_{\text{rsr}}) = q_{\text{rsr}} - \pi.
\]

Proposition 19. For class \( a_{31} \) (i.e. \( \pi < \alpha \leq 3\pi/2 \), \( 0 < \beta \leq \pi/2 \)), the optimal solution is LSL if \( S_{31} < 0 \), and it is LSR otherwise, where

\[
S_{31}(q_{\text{lsl}}) = q_{\text{lsl}} - \pi.
\]

Proposition 20. For class \( a_{42} \) (i.e. \( 3\pi/2 < \alpha \leq 2\pi \), \( \pi/2 < \beta \leq \pi \)), the optimal solution is RSL if \( S_{42} < 0 \), and it is LSL otherwise, where

\[
S_{42}(t_{\text{lsl}}) = t_{\text{lsl}} - \pi.
\]

4.2.4. Equivalency group \( \{ a_{14}, a_{41} \} \)

According to Proposition 4, path classes \( a_{14} \simeq a_{41} \) are in the same equivalency group, and so, by the rules established in the transformation theorem, the classification of optimal solutions for class \( a_{14} \) leads to the optimal solutions for class \( a_{41} \).

Proposition 21. For the long path case, the optimal solution corresponding to the element \( a_{14} \) is \( \{ \text{RSR or LSR or RSL} \} \).
thus defines the switch condition for the optimal path from LSR to RSR. Similarly, the critical angle $\tilde{\beta}$ indicates the switch of the optimal path from RSR to RSL.

The proof of these facts is similar to those for classes $a_{11}$ and $a_{13}$. Define two critical angles: $\alpha = \tilde{\alpha}(d, \beta)$ — the angle at which paths LSR and RSR degenerate into the path SR, and $\beta = \tilde{\beta}(d, \alpha)$ — the angle at which paths RSR and RSL degenerate into the path RS. First consider the case when $\alpha \in [\tilde{\alpha}, \pi/2]$ and $\beta \in [3\pi/2, \beta(\tilde{\alpha})]$. It is claimed that if $\alpha \in [\tilde{\alpha}, \pi/2]$ and $\beta \in [3\pi/2, \beta(\tilde{\alpha})]$ then the upper bound for $L_{rsr}$ is limited by

$$\max_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [3\pi/2, \beta(\tilde{\alpha})]} L_{rsr} \leq d - 2 + \pi.$$  

Indeed, the gradient of $L_{rsr}$ is strictly positive when $\alpha \in [\tilde{\alpha}, \pi/2]$ and strictly negative when $\beta \in [3\pi/2, \tilde{\beta}]$. In the regions $\alpha \in [\tilde{\alpha}, \pi/2]$ and $\beta \in [3\pi/2, \tilde{\beta}]$, the maximum length of path RSR occurs when $D = 2$ and $D_3 = 2$ (this follows from monotonicity of the function $L_{rsr}$), i.e. when

$$\min_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [\tilde{\beta}, 2\pi]} L_{rsr} \geq \sqrt{d^2 + 2d + 3\pi/2}.  \quad (30)$$

One can conclude that if $\alpha \in [\tilde{\alpha}, \pi/2]$ and $\beta \in [3\pi/2, \tilde{\beta}]$ then the lower bound on $L_{lsr}, L_{rls}$, and $L_{lsr}$ is

$$\min_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [3\pi/2, \tilde{\beta}]} \{L_{lsr}, L_{rls}, L_{lsr}\} \geq 3\pi + d + 2, \quad (30)$$

$$\min_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [3\pi/2, \tilde{\beta}]} L_{rls} \geq 3\pi/2 + \sqrt{d^2 + 2d}, \quad (31)$$

$$\min_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [3\pi/2, \tilde{\beta}]} L_{lsr} \geq 3\pi/2 + \sqrt{d^2 + 2d}. \quad (32)$$

This leads to the condition

$$\min_{\alpha \in [\tilde{\alpha}, \pi/2], \beta \in [3\pi/2, \tilde{\beta}]} \{L_{lsr}, L_{rls}, L_{lsr}\} \geq 3\pi/2 + \sqrt{d^2 + 2d}. \quad (33)$$

Notice that the lower bound on paths $\{L_{lsr}, L_{rls}, L_{lsr}\}$ is larger than the upper bound on $L_{rsr}$ (Eqs. (29) and (33)). This completes the proof of the fact that if $\alpha \in [\tilde{\alpha}, \pi/2]$ and $\beta \in [3\pi/2, \beta(\tilde{\alpha})]$, then the path RSR is the optimal
solution. For class $a_{14}$ two more candidates are possible, i.e. LSR and RSL. The proposition below relates the set of candidates for class $a_{14}$ to the accompanying class $a_{41}$.

**Proposition 22.** Since $a_{14} \preceq a_{41}$ and the optimal solution for class $a_{14}$ is \{LSR or RSL or RSR\}, the optimal solution for class $a_{41}$ is \{RSL or LSR or LSL\}.

Similar to the class $a_{13}$, the critical angles $\tilde{\alpha}$ and $\tilde{\beta}$ define the switch of the optimal solution from RSR to LSR and from RSR to RSL, respectively. To simplify the calculations, we choose to build our classification based on the length of segments $t_{rsr}$ and $q_{rsr}$. Following the same logic as in the case of class $a_{13}$, one notices that $t_{rsr}$ and $q_{rsr}$ cannot exceed $\pi$ if RSR is an optimal solution for class $a_{14}$. If either of the two segments $t_{rsr}$ or $q_{rsr}$ is bigger than $\pi$, then there is always another path that is shorter than RSR; i.e. if $t_{rsr} > \pi$ then the optimal solution is LSR, and if $q_{rsr} > \pi$ then the optimal solution is RSL. This observation leads to the following classification rule.

**Proposition 23.** For class $a_{14}$ (i.e. $0 < \alpha \leq \pi/2$, $3\pi/2 < \beta \leq 2\pi$), if

1. $S_{14}^1(t_{rsr}) > 0$, then the optimal solution is RSR,
2. $S_{14}^2(q_{rsr}) > 0$, then the optimal solution is RSL,
3. if neither (1) or (2) holds, then the optimal solution is RSR,

where

$$S_{14}^1(t_{rsr}) = t_{rsr} - \pi,$$
$$S_{14}^2(q_{rsr}) = q_{rsr} - \pi.$$  \hspace{1cm} (34)

By applying the transformation theorem, obtain the switching conditions for class $a_{41}$.

**Proposition 24.** For class $a_{41}$ (i.e. $3\pi/2 < \alpha \leq 2\pi$, $0 < \beta \leq \pi/2$), if

1. $S_{41}^1(t_{rsr}) > 0$, then the optimal solution is RSL,
2. \(S_{14}^2(q_{rsr}) > 0\), then the optimal solution is LSR.
3. if neither (1) or (2) holds, then the optimal solution is LSL,

where
\[
S_{14}^1(t_{isl}) = t_{isl} - \pi, \tag{36}
\]
\[
S_{14}^2(t_{isl}) = q_{isl} - \pi. \tag{37}
\]

4.2.5. Equivalency group \( \{a_{22}, a_{33} \} \)

According to Proposition 4, classes \(a_{22} \simeq a_{33}\) are in the same equivalency group, and so the classification of optimal solutions for class \(a_{22}\) leads to that for class \(a_{33}\).

Proposition 25. For the long path case, the optimal solution corresponding to the class \(a_{22}\) is \{LSL or RSL or RSR\}.

The path LSR is excluded from the consideration since \(L_{isl}\) is always larger than \(L_{rsl}\). Indeed, for \(\alpha = \beta = \pi\) the lengths \(L_{isl} = L_{rsl}\), and path LSR has its minimum at \(\alpha = \beta = \pi\), and path RSL—its maximum within class \(a_{22}\).

Proposition 26. Since \(a_{22} \simeq a_{33}\) and the optimal solution for \(a_{22}\) is \(\{LSL or RSL or RSR\}\), then the optimal solution for \(a_{33}\) is \(\{RSR or LSL or LSR\}\).

The following proposition gives the condition for finding the optimal paths.

Proposition 27. For class \(a_{22}\) (i.e. \(\pi/2 < \alpha \leq \pi, \pi/2 < \beta \leq \pi\)), if
1. \(\alpha > \beta\) and \(S_{12}^1 < 0\), the optimal solution is LSL,
2. \(\alpha > \beta\) and \(S_{12}^1 > 0\), the optimal solution is RSL,
3. \(\alpha < \beta\) and \(S_{22}^2 < 0\), the optimal solution is RSR,
4. \(\alpha < \beta\) and \(S_{22}^2 > 0\), the optimal solution is RSL,

where the switching functions are
\[
S_{12}^1(p_{isl}, p_{rsr}, t_{rsl}) = p_{isl} - p_{rsr} - 2(t_{rsl} - \pi), \tag{38}
\]
\[
S_{12}^2(p_{rsr}, p_{isl}, q_{rsl}) = p_{rsr} - p_{isl} - 2(q_{rsl} - \pi), \tag{39}
\]
and \(p_{isl}, p_{rsr}, p_{isl}, t_{rsl},\) and \(q_{rsl}\) are defined by (3) and (9), respectively.

The proof of the proposition is similar to that for class \(a_{12}\) with the only difference that the set of candidates depends upon the relations between \(\alpha\) and \(\beta\). Notice that if \(\alpha = \beta\) then the length of path LSL is equal to that of RSR. When \(\alpha\) starts increasing, path LSL becomes shorter than path RSR (see Fig. 11). Similarly, using the transformation theorem, obtain the solutions for class \(a_{33}\) are as follows.

Proposition 28. For class \(a_{33}\) (i.e. \(\pi < \alpha \leq 3\pi/2, \pi < \beta \leq 3\pi/2\)), if
1. \(\alpha < \beta\) and \(S_{13}^1 < 0\), the optimal solution is RSR,
2. \(\alpha > \beta\) and \(S_{13}^1 > 0\), the optimal solution is LSL,
3. \(\alpha < \beta\) and \(S_{23}^2 < 0\), the optimal solution is LSL,
4. \(\alpha > \beta\) and \(S_{23}^2 > 0\), the optimal solution is LSR,

where the switching functions are
\[
S_{13}^2(p_{rsl}, p_{isl}, t_{isl}) = p_{rsl} - p_{isl} - 2(t_{isl} - \pi), \tag{40}
\]
\[
S_{13}^2(p_{isl}, p_{rsl}, q_{isl}) = p_{isl} - p_{rsl} - 2(q_{isl} - \pi). \tag{41}
\]
4.2.6. Equivalency group \( \{a_{23}, a_{32}\} \)

According to Proposition 4, classes \( a_{23} \approx a_{32} \) are in the same equivalency group. The corresponding classification of optimal solutions proceeds as follows.

**Proposition 29.** For the long path case, the optimal solution corresponding to the class \( a_{23} \) is RSR.

Four path candidates need be considered here — RSR, RSL, LSR, LSL. Note that

\[
\max_{\alpha \in [\pi/2, \pi], \beta \in [\pi, 3\pi/2]} \mathcal{L}_{\text{rsr}} = \min_{\alpha \in [\pi/2, \pi], \beta \in [\pi, 3\pi/2]} \{\mathcal{L}_{\text{rsl}} \mathcal{L}_{\text{lsl}} \mathcal{L}_{\text{lsr}}\}.
\]

The maximum of the length of path RSR and the minimum of \( \{\mathcal{L}_{\text{rsl}} \mathcal{L}_{\text{lsr}} \mathcal{L}_{\text{lsl}}\} \) occur when \( \alpha = \beta = \pi \). By applying the transformation theorem, the classification for \( a_{32} \) is given by the following proposition.

**Proposition 30.** Since \( a_{23} \approx a_{32} \) and the optimal solution for \( a_{23} \) is RSR, the optimal solution for \( a_{32} \) is LSL.

The switching functions are logical functions of Boolean type. They are not uniquely defined. The choice of a particular function will be usually guided by computational considerations.

5. The main result

We can now summarize the whole scheme developed above, which presents the main result of this work. Using the scheme, the problem of finding the shortest smooth path between two configurations is solved without an explicit calculation of the paths involved. Instead, a simple logical scheme is used based on the aggregation of all possible paths into classes \( a_{ij} \) and on the equivalency groups as defined above.

The input to the scheme are the angular quadrants of the directional angles \((\alpha, \beta)\); its output is the name of the element of the Dubins set that presents the shortest path. The scheme forms a decision tree summarized in the table in Fig. 12. Each block of the table represents one element \( a_{ij} \) of matrix \( \{a_{ij}\} \) and includes the corresponding options for the optimal path. It may take only one step to obtain the solution, as e.g. the solution RSL for the quadrants \((1, 1)\), or it may take two steps, as for the quadrants \((3, 1)\), or it may take at most three steps, as for the quadrants \((2, 2)\) (Fig. 12). When the second and third steps are necessary, their outcome is determined by the signs of one or two switching functions \( S \). The general form of functions \( S \) is

\[
f(s_1, s_2, s_3) = s_1 - s_2 - 2(s_3 - \pi), \quad g(s) = s - \pi.
\]
The complete list of switching functions for all classes $a_{ij}$ is as follows (note that for some classes, the unique optimal solution is obtained directly, without switching functions):

Class $a_{11}$: unique solution,
Class $a_{12}$: $S_{12} = f(p_{rsr}, p_{rsl}, q_{rlsl}) = p_{rsr} - p_{rsl} - 2(q_{rlsl} - \pi)$,
Class $a_{13}$: $S_{13} = g(t_{rsr}) = t_{rsr} - \pi$,
Class $a_{14}$: $S_{14}^1 = g(t_{rsr}) = t_{rsr} - \pi$, \hspace{1cm} $S_{14}^2 = g(q_{rsr}) = q_{rsr} - \pi$,
Class $a_{21}$: $S_{21} = f(p_{lsl}, p_{rsl}, t_{rlsl}) = p_{lsl} - p_{rsl} - 2(t_{rlsl} - \pi)$,
Class $a_{22}$: if $\alpha > \beta$, then $S_{22}^1 = f(p_{ldl}, p_{lsl}, t_{ldl}) = p_{ldl} - p_{lsl} - 2(t_{ldl} - \pi)$,
Class $a_{23}$: unique solution,
Class $a_{24}$: $S_{24} = g(q_{rsr}) = q_{rsr} - \pi$,
Class $a_{31}$: $S_{31} = g(q_{ldl}) = q_{ldl} - \pi$,
Class $a_{32}$: unique solution,
Class $a_{33}$: if $\alpha < \beta$, then $S_{33}^1 = f(p_{rsr}, p_{lsl}, t_{ldl}) = p_{rsr} - p_{lsl} - 2(t_{ldl} - \pi)$,
Class $a_{34}$: if $\alpha > \beta$, then $S_{34}^2 = f(p_{ldl}, p_{lsl}, q_{ldl}) = p_{ldl} - p_{lsl} - 2(q_{ldl} - \pi)$,
Class $a_{41}$: $S_{41} = f(p_{lsl}, p_{lsl}, t_{ldl}) = p_{lsl} - p_{lsl} - 2(t_{ldl} - \pi)$,
Class $a_{42}$: $S_{42} = g(t_{lsl}) = t_{lsl} - \pi$,
Class $a_{43}$: $S_{43} = f(p_{lsl}, p_{lsl}, q_{ldl}) = p_{lsl} - p_{lsl} - 2(q_{ldl} - \pi)$,
Class $a_{44}$: unique solution.

<table>
<thead>
<tr>
<th>Final Quadrant</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Quadrant</td>
<td>RSL</td>
<td>if $S_{11} &lt; 0$ then RSR, \hspace{1cm} if $S_{11} &gt; 0$ then LSR</td>
<td>if $S_{13} &lt; 0$ then RSR, \hspace{1cm} if $S_{13} &gt; 0$ then LSR</td>
<td>if $S_{14} &gt; 0$ then LSR, \hspace{1cm} if $S_{14} &lt; 0$ then RSR</td>
</tr>
<tr>
<td>2</td>
<td>if $S_{21} &lt; 0$ then LSL, \hspace{1cm} if $S_{21} &gt; 0$ then LSR</td>
<td>if $S_{23} &lt; 0$ then RSR, \hspace{1cm} if $S_{23} &gt; 0$ then LSR</td>
<td>RSR</td>
<td>if $S_{24} &lt; 0$ then LSR, \hspace{1cm} if $S_{24} &gt; 0$ then RSR</td>
</tr>
<tr>
<td>3</td>
<td>if $S_{31} &lt; 0$ then LSL, \hspace{1cm} if $S_{31} &gt; 0$ then LSR</td>
<td>LSL</td>
<td>if $S_{33} &lt; 0$ then RSR, \hspace{1cm} if $S_{33} &gt; 0$ then LSR</td>
<td>if $S_{34} &lt; 0$ then LSR, \hspace{1cm} if $S_{34} &gt; 0$ then RSR</td>
</tr>
<tr>
<td>4</td>
<td>if $S_{41} &gt; 0$ then RSR, \hspace{1cm} if $S_{41} &lt; 0$ then LSR</td>
<td>if $S_{43} &lt; 0$ then LSR, \hspace{1cm} if $S_{43} &gt; 0$ then RSR</td>
<td>LSR</td>
<td>if $S_{44} &lt; 0$ then LSR, \hspace{1cm} if $S_{44} &gt; 0$ then RSR</td>
</tr>
</tbody>
</table>

Fig. 12. The decision table for finding the shortest path.
Take, e.g. a set \( \alpha \in [0, \pi/2], \beta \in [\pi/2, \pi] \), which corresponds to the element \( a_{12} \) of matrix \( \{a_{ij}\} \). Block (1, 2) in the table in Fig. 12 suggests that the optimal solution is either RSL or RSR. The switching function \( S_{12} \) listed in block (1, 2) is then calculated as in the list above (Eq. (42)); its elements \( p_{rsl}, p_{rsr} \) and \( q_{rsl} \) come from the second equation of (5) and the second and third equations of (9), respectively. The sign of \( S_{12} \) then determines which of the two versions indicated in block (1, 2) is the unique solution.

To demonstrate efficiency of the proposed classification scheme, consider the task of finding the shortest smooth path from \( P_i \) to \( P_f \), located on the distance \( |P_i P_f| = 6 \), such that the path starts and ends with the directions of motion \( \alpha = \pi/6 \) and \( \beta = \pi/3 \). The path radius of curvature is limited by \( \rho = 1 \). According to Dubins result, six paths candidates \{LSL, RSR, RSL, LSR, RLR, LRL\} have to be calculated and compared. Using expressions given in Section 2

\[ L_{isl} = t_{isl} + p_{isl} + q_{isl} = -\alpha + \beta + p_{isl} = 12.4526, \]
\[ L_{rsr} = t_{rsr} + p_{rsr} + q_{rsr} = \alpha - \beta + p_{rsr} = 12.1361, \]
\[ L_{lsl} = t_{lsl} + p_{lsl} + q_{lsl} = \alpha - \beta + 2t_{lsl} + p_{lsl} = 18.3890, \]
\[ L_{rsl} = t_{rsl} + p_{rsl} + q_{rsl} = -\alpha + \beta + 2t_{rsl} + p_{rsl} = 6.2488, \]
\[ L_{lrl} = t_{lrl} + p_{lrl} + q_{lrl} = -\alpha + \beta + 2p_{lrl} \quad \text{(path is not feasible)}, \]
\[ L_{rsl} = t_{rsl} + p_{rsl} + q_{rsl} = -\alpha + \beta + 2p_{rsl} \quad \text{(path is not feasible)}. \]

After all paths-candidates are calculated and compared, we conclude that the shortest path is RSL. In contrast, if we use the approach proposed in this paper, then based only on information about the initial and final configuration and without any calculations involved, we can make the same conclusion. Indeed, for the case studied, the initial and final configurations are in the first quadrant, i.e. the path belongs to the class \( a_{11} \). According to the look-up table (Fig. 12), the optimal solution for all paths from this class have to have the topology RSL. Thus, using the classification scheme we were able to obtain the optimal solution without any calculations involved. This example is a good illustration of the computational efficiency of the proposed classification scheme.

6. Conclusion

The central idea developed in this work is that the problem of finding the shortest path between two configurations can be reduced to a logical manipulation of the set of appropriate path candidates, without their explicit calculation. This is in sharp departure from the direct computation and comparison of the candidate paths that the existing techniques require. The candidate paths come from the sufficient set known as the Dubins set [5]. One direct benefit of the suggested scheme is computational savings — an important consideration in real-time control. For example, when attempting to find the shortest path to a given position/orientation (configuration) for a driverless car or a mobile robot, one would simply find in the table (Fig. 12), the element that corresponds to the initial and final configurations, and then pinpoint the unique solution either immediately or using the sign of an appropriate switching function of the form (42).

As mentioned in Section 1, the derivation of the approach is simplified if the problem at hand is divided into two cases, called here the long path case and the short path case. To save space, the suggested logical classification scheme is fully developed here only for the long path case. Situations with short paths require a roughly similar, though a bit tedious, analysis, resulting in a computational procedure that is somewhat more complex and less economical than the one presented here (see Proposition 5 for a formal definition of both cases).

The presented result also gives a new interesting insight into the nature of Dubins’ problem. It suggests that partitioning of the appropriate C-space can be a powerful tool for analyzing the shortest path problem in more general and complex cases, as e.g. in finding the shortest path between a point and a manifold.
References


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