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On Optimal Nonholonomic Paths in a Limited Space

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Abstract

This work addresses the problem of finding the shortest path for a vehicle (say, a mobile robot or a car) moving in a limited workspace. The proposed approach makes use of a tool dubbed the Reflective Unfolding operator which has a clear geometric interpretation and provides an interesting means for solving other trajectory design problems. Our previosly reported result [1], which relates to the special case of a car maneuvering within a disc-shaped area, is extended here to the general case of an arbitrarily shaped area. The approach is illustrated by computer simulations.

1 Introduction

We pose the following two questions (see Fig. 1): Given two points, start and target, within a closed planar area $\mathbf{W} \subset \mathbf{R}^2$, each with a prescribed direction of motion in it, and assuming a possibility of reversals of motion, (i) what is the shortest path of bounded curvature that connects the points and lies completely in $\mathbf{W}$? (ii) what is the minimum number of motion reversals (path cusps) one needs to arrive at the target point with the prescribed orientation? This kind of questions appear in various applications with nonholonomic motion constraints, such as in motion planning for driverless cars.

The proof of existence of a path between any two configurations lying in the same workspace was shown by J.P. Laumond [2]. The idea of the proof is to approximate the path by a sequence of short back and forth motions (reversals, cusps). As shown in [3], such approximations may produce long paths, perhaps with very many reversals. It would therefore be of interest to attempt to find the "best" motion - one that minimizes the path length and the number of reversals.

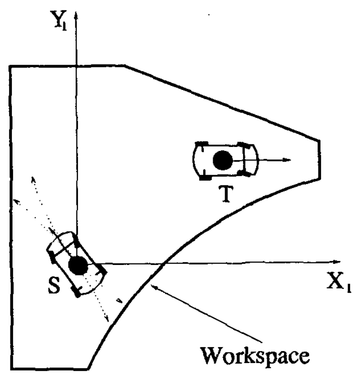

Figure 1: From its initial configuration $S = (q_0, \phi_0)$, the car is to arrive at the final configuration $T = (q_f, \phi_f)$, along the path of the shortest length possible and of the lowest complexity.

For the case with unlimited operational space, Dubins [4] showed how to compute the shortest smooth path, with no motion reversals. The Reeds and Shepp work [5] presents an extension of this result by Dubins to the more complex case with cusps. A case important for applications (e.g. mobile robots on the factory floor, or driverless, or teleoperated transportation systems) involves planning motion with reversals in a constrained environment. This general case of an optimal path in a limited space, left open by the works above, is the subject of this paper.

The text below is organized as follows. The problem statement and necessary definitions are given in Section 2. The transformation and the RU operator that form the basis of the proposed approach appear in Section 3. The overall strategy for designing the path in the workspace $\mathbf{W}$ is derived in Section 4; its convergence properties are discussed in Section 5, and the optimality of solutions that it produces is discussed in Section 6. The resulting algorithm is summarized in Section 7. An illustration of the algorithm performance appears in Section 8. A summary and discussion of the obtained results in Section 9 completes the paper. The details that have been left out due to lack of space can be found in [6].

2 The Approach

We introduce a new tool called the *Reflective Unfolding operator* (*RU* operator - see Section 3) which maps the original problem of planning a path with reversals in the domain $\mathbf{W}$ into an equivalent problem of planning a smooth cusp-less path in an unlimited space. A successive application of the RU operator yields the optimal solution with reversals in the original limited space.

The idea is as follows. Assume for the moment that the path connecting the initial configuration and the final configuration does already exist and consists of n arcs, each of radius ρ_{min}, and thus of $n-1$ cusps. Order all cusps sequentially, starting with the initial configuration. In a single application of the RU operator, it keeps the first of two arcs adjoining the first cusp, and "unfolds" the second arc so as to produce a smooth cuspless piece of circle, while preserving the original tangent to both arc segments, Figure 2. The next cusp is then treated in a similar fashion, and so on, eventually transforming the set of arc segments into a large circular arc $C = (O, \rho_{min})$ of radius ρ_{min} centered at some point O, with multiple copies of the domain $\mathbf{W}$ superimposed on it (see Section 4 and Figure 4). Once the process reaches the final configuration, one only needs to fold all the arc segments back into the $\mathbf{W}$ – and the actual path is complete.

The RU operator possesses a number of properties that make it a good tool for calculating optimal paths. For example, since the car's initial and final orientations define uniquily the corresponding tangent lines to the circle $C = (O, \rho_{min})$, by measuring the distance along the circle C between those two configurations one can quickly calculate the number of segments – and, therefore, the number of cusps and the total length – in the optimal path, even without calculating the actual path.

3 Reflective Unfolding Operator

The Reflective Unfolding operator, or RU operator, presents the main tool for solving the problem of

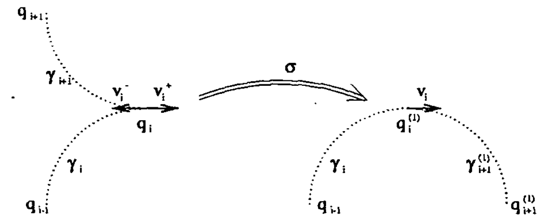

Figure 2: The RU operator, σ, eliminates a cusp at the carrier q_{i+1} and maps segment γ_{i+1} into segment $\gamma_{i+1}^{(1)}$, formaly, $\sigma_{q_{i+1}} : \gamma_i \uplus_{q_{i+1}} \gamma_{i+1} \to \gamma_i \uplus \gamma_{i+1}^{(1)}$.

motion in a constrained environment. The term *motion flow* below uniquely defines in a parametric form the configuration, position and orientation, of a nonholonomic system. Consider a motion flow consisting of two segments, $\gamma_i(t)$ and $\gamma_{i+1}(t)$, Figure 2. As parameter t increases, the flow continues from segment $\gamma_i(t)$ to segment $\gamma_{i+1}(t)$. Note different orientations at both segments: when switching from $\gamma_i(t)$ to $\gamma_{i+1}(t)$, the car reverses the direction of motion.

4 Control strategy and dual problem

We are now ready to formulate the control strategy for generating the shortest path of bounded curvature within a limited workspace. First, the control strategy will be defined, and then it will be proven to be the optimal strategy. The proof is based on the analysis of duality of the original problem in the domain $\mathbf{W}$ and the equivalent problem in the chain of domains $\mathbf{W_1}, \mathbf{W_2}, \mathbf{W_3}, \ldots$. The transformation from the original problem to its dual is obtained by sequentially applying the RU operator.

Control strategy. Assume that the closed domain $\mathbf{W} \subset \mathbf{R}^2$ in which the car operates is small enough, so that the assumption of an unlimited space necessary for obtaining Reeds-Shepp solutions [5] does not apply. To solve the problem of maneuvering within $\mathbf{W}$, a natural recipe to follow would be to (1) move along path segments of maximum curvature (arcs of circles of radius ρ_{min}), and (2) use most effectively the free space available – that is, try to extend each path segment up to the boundary ·f $\mathbf{W}$; we will call this a boundary-to-boundary strategy.

That is, the car should move (forward or backward) along an arc of radius ρ_{min} to the boundary of $\mathbf{W}$, then make a reversal, move along another arc of radius ρ_{min} until the boundary is reached again, and so on. Intuitively, the boundary-to-boundary strat-

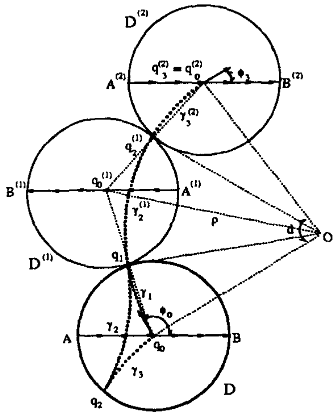

Figure 3: The mapping is defined as a central symmetry with respect to the reflection point. Here, the central symmetry with respect to point q_1 maps point q_2 into $q_2^{(1)}$, A into $A^{(1)}$, and B into $B^{(1)}$; disc D is mapped into the disc $D^{(1)}$. The motion flow $q_0 \to q_1 \to q_2 \to q_0$ within the disc $D(q_0, R)$ is equivalent to the flow $q_0 \to q_1 \to q_2^{(1)} \to q_0^{(2)}$ that corresponds to the arc of radius ρ_{min} centered at O.

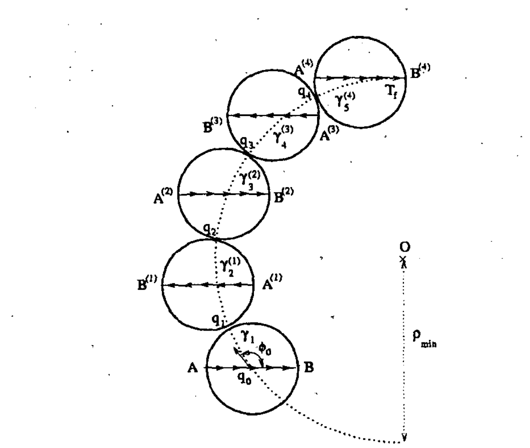

Figure 4: Motion with reversals within the domain $\mathbf{W} = D(q_0, R)$ is equivalent to the motion along a circle of radius ρ_{min} centered at point O.

egy should give the fastest possible convergence to the car's desired final orientation (this fact will be proven in the next section). However, the question is how to construct a strategy that will result in the final *configuration* (position and orientation). Another part of our goal is to produce the shortest path possible. The algorithm for choosing the points of reversal of motion is discussed below.

The dual problems For the purpose of illustration, consider the workspace $\mathbf{W}$ to be a disc D of some radius R; the car's initial position q_0 is at the center O of D; the car's final orientation is horizontal. (The general case, for an arbitrarily-shaped workspace $\mathbf{W}$ and arbitrary initial/final positions/orientations is discussed in Section 8). A typical path in the disc $D(q_0, R)$ is shown in Figure 3. Starting at the configuration $p_0 = (q_0, \phi_0)$, the car first moves along an arc of radius ρ_{min} and reaches the boundary of $\mathbf{W}$ at point $q_1 \in \partial\mathbf{W}$, with orientation ϕ_1^+ (equivalently, with the unit velocity vector v_1^+). Then the car reverses and moves backward along the second arc segment, with the initial velocity vector v_1^-, reaching $\partial\mathbf{W}$ at point q_2, with orientation v_2^+. The path from $p_0 = (q_0, \phi_0)$ to $p_2 = (q_2, \phi_2)$ thus contains two arcs, γ_1 and γ_2, connected by a cusp with the common tangent at q_1 (Figure 3).

By applying mapping (the RU operator) σ_{q_1} to the path $\gamma_1 \uplus_{q_1} \gamma_2$, a smooth subpath $\gamma_1 \uplus \gamma_2^{(1)}$ is obtained. Similarly, by applying mapping σ_{q_2} to subpaths $\gamma_1 \uplus \gamma_2^{(1)}$ and $\gamma_3^{(1)}$ connected by cusp $q_2^{(1)}$, obtain the subpath $\gamma_1 \uplus \gamma_2^{(1)} \uplus \gamma_3^{(2)}$. That is, the RU operator maps the boundary-to-boundary motion flow within a disc into an equivalent motion flow along a circle and without cusps.

Note that the equivalent problem is tantamount to covering the equivalent path by discs of radius R following the mapping rules. Once this is done, the arc segments of the equivalent path can be "folded back" into the actual workspace $\mathbf{W} = D(q_0, R)$ to complete the construction of the sought path. Below the equivalent problem is studied in more detail.

5 Convergence to final configuration

Here we develop the control strategy which makes use of the composite transformation process described above, and guarantees that the car reaches its final configuration from its initial configuration, while keeping the path within the domain $\mathbf{W}$. The corresponding procedures are developed below: the orientation alignment; position alignment; boundary-to-boundary motion; and calculation of the nonstandard finishing maneuvers inside the domain $\mathbf{W}$.

Orientation alignment. Observe that limiting path segments to the arcs of radius ρ_{min} is the fastest way to reach the final orientation. Indeed, assuming a unit velocity, the orientation changes as

$$\Delta\phi = t/\rho \qquad (1)$$

where $\Delta\phi$ is the change in the orientation angle, t – a parameter of length, and ρ – radius of the arc. Equa-

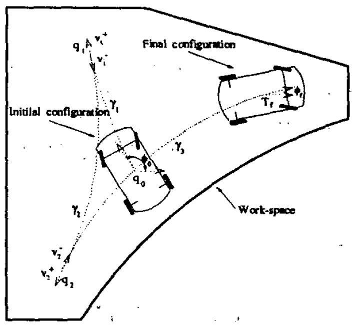

Figure 5: This path, from the car's initial configuration (q_0, ϕ_0) to its final configuration (q_f, ϕ_f) requires two reversals of motion. At points q_1 and q_2 the car touches the workspace boundary.

tion (1) simply states that, with $\Delta\phi$ fixed, the shortest path is obtained when the radius of curvature is minimum. That is, if the car moves along a circle of larger radius, the angle between the vector of velocity and the initial orientation changes slower than it would along a circle of smaller radius.

Position alignment. The shortest possible path connecting two configurations can be simply calculated based on the difference between the initial and final orientation angles:

$$Length(\gamma(t)) = \rho_{min} \cdot |\phi_f - \phi_0| \qquad (2)$$

This gives the lower bound on all possible paths connecting configurations $p_0 = (q_0, \phi_0)$ and $p_f = (q_f, \phi_f)$. The question now is whether this lower bound is achievable – or equivalently, whether there exists a control strategy that would brings the car to the final configuration while delivering the path of length (2).

6 Optimality of the control strategy

Proposition 1 *The motion flow generated by the proposed control strategy is a geodesic flow.*

The proof of this proposition makes use of the following geometrically obvious auxiliary statement (see Figure 6):

Proposition 2 (auxiliary) *Let η_1 and η_2 be two circular arcs of radii R_1 and R_2, respectively. Suppose η_1 connects some initial configuration (M_0, α_0) with the final configuration (M_1, α_1), and η_2 connects the same initial configuration (M_0, α_0) with the final configuration (M_2, α_2). Then, if $R_2 > R_1$ and the final orientations are the same, $\alpha_1 = \alpha_2$, then arc η_2 is longer than arc η_1.*

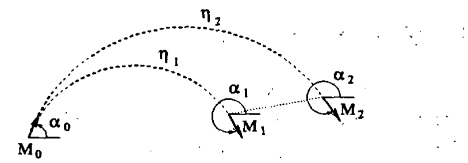

Figure 6: If $\alpha_1 = \alpha_2$, and η_1 and η_2 are circular arcs of the radii R_1 and R_2, and $R_2 > R_1$, then arc η_2 is longer than arc η_1

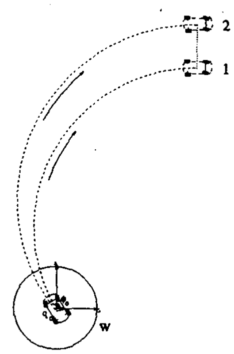

Figure 7: This example illustrates that the optimal solution is to move along a path of maximum curvature (smallest radius); both paths achieve the final orientation, but path "1" is shorter than "2".

According to the control strategy, the car moves along a path of maximum curvature, $1/\rho_{min}$. For the equivalent problem this corresponds to the motion along a circle of radius ρ_{min}. Assume this is path "1" in Figure 7. The task is complete when the car arrives at the final configuration with zero angle orientation. In the equivalent problem, this corresponds to arriving at the top point on circle $C(O, \rho_{min})$, where the tangent to C is horizontal.

There are infinitely many paths that would bring the car from its initial to the final orientation (path "2" in Figure 7 is one example). According to the equivalent problem setup, all points of those other paths will appear outside the circle $C(0, \rho_{min})$. By Proposition 2, any path outside circle C has to be longer. In the previous section it was also shown that if the radius of curvature is larger or equal to the workspace diameter, one can always construct a snake connecting the initial and final configurations. This completes the proof of Proposition 1 – that the path produced by the proposed algorithm is the shortest possible path.

7 The Algorithm

The algorithm consists of two procedures: ***Initialization***, which sets up system parameters, and ***Main Body***. The latter, in turn, makes use of procedures ***Unfolding*** (which realizes the Reflective Unfolding operator) and ***Finishing Maneuvers*** (which handles the last two maneuvers to complete the task). An example of the algorithm performance will be considered in Section 8. Assume that the initial and final configurations are given.

Initialization:

— Initialize workspace **W**. Based on sensory data, construct the occupancy grid or map.

— Construct circle $C(O, \rho_{min})$ (Section 4) tangent to the initial orientation. Choose the origin of system $\{XY\}$ at center O of $C(O, \rho_{min})$, and X-axis to coincide with the final orientation.

— Set the maneuvers counter to $i = 1$.

Main Body: The procedure is executed at each step; it operates with respect to the world coordinate system $\{XY\}$. It uses two arc segments: $\gamma_i(x, y)$, the trimmed arc for step i, and $\gamma_f(x, y)$, the trimmed arc portion between the final position and the point on **W** boundary where the arc originates:

If $T_i = T_f$, stop.
Find $\gamma_i(x, y)$; Find $\sum_{j=1}^{i} Length(\gamma_j(x, y))$.
If
$\sum_{j=1}^{i} Length(\gamma_j(x, y)) + Length(\gamma_f(t)) < Length(\gamma(t))$
Then call *Unfolding*;
Else call *Finishing Maneuvers*;
Iterate $i = i + 1$.

Unfolding: The procedure operates on two arc segments, $\gamma_i(x, y)$ and $\gamma_{i+1}(x, y)$, with respect to the cusp q_i in the frame X_iY_i:

— Find q_i (intersection of $\gamma_i(x, y)$ and $C(O, \rho_{min})$) with respect to {X Y};

— Project q_i onto system X_iY_i;

— Apply the centro-symmetric transformation to $\mathbf{W}_i$ with respect to q_i;

— Describe $\mathbf{W}_{i+1}$ analytically;

— Transform $\mathbf{W}_{i+1}$ to $\{XY\}$;

— Find $\gamma_{i+1}(x, y)$ (intersection of $C(O, \rho)$ and $\mathbf{W}_{i+1}$;

— Return $\gamma_{i+1}(x, y)$.

Finishing Maneuvers. The procedure is called by *Main Body* when condition $\sum_{j=1}^{i} Length(\gamma_j(x, y)) - Length(\gamma_f(t)) < Length(\gamma(t))$ is not satisfied (which happens when $\mathbf{W}_i \bigcap \mathbf{W}_f \neq \emptyset$). It uses the final position on i-th iteration, $T_i \in \mathbf{W}_i$, and the final position $T_f \in \mathbf{W}_f$. It returns to *Main Body* the final configuration:

— Define vector $\overrightarrow{T_iT_f}$;

— Define chord $[q_{i+1}\ q_{i+2}] \in C(O, \rho_{min})$;

— Call *Unfolding*, with q_{i+1} as input;

— Call *Unfolding*, with q_{i+2} as input.

Remark: The resulting sequence of maneuvers is recorded in the seqience $\{q\}_i$, where $q_1, \cdots, q_n$ are reflection (cusp) points. An illustration of the algorithm performance for the general case is given in Section 8.

8 Example

The general algorithm (Section 7) is illustrated here on a simulated example, shown in Figure 1. For convenience, the final orientation is taken along the horizontal line (which is always possible by properly rotating **W**). There are two important differences compared to the special case discussed in [1]: (a) from its initial configuration the car is required to arrive at the different, both in position and orientation, final configuration (positions S and T in Figure 1); (b) the workspace is of some arbitrary shape.

The procedure presents a sequence of geometric constructs leading to the overall solution obtained in a closed form.

The task thus is: given the car's initial and final configurations in its workspace **W**, Figure 1, find the shortest possible path for the car's center of mass that lies fully inside **W**, subject to the restriction on the path radius of curvature, $\rho = \rho_{min}$. Use Figure 8 to follow the solution process.

Step 1: Construct a circle of radius ρ_{min} tangent to the car's initial orientation vector.

Step 2: Copy domain **W** by translating it so that point T coincides with point T_f; this produces $\mathbf{W}_f$ (in Figure 8, $\mathbf{W}_f = \mathbf{W}_7$).

Step 3: Going back to $\mathbf{W}_1$, apply the central symmetry (RU operator) with respect to the point where the boundary $\partial\mathbf{W}$ intersects the virtual path, obtaining the workspace prototype $\mathbf{W}_2$.

Step 4: Repeat Step 3 few more times until the current prototype overlaps with $\mathbf{W}_f$ (in Figure 8 this will take three more reflections, ending with $\mathbf{W}_5$.

Step 5: Draw a line segment between points T_5 in $\mathbf{W}_5$ and $T_7 = T_f$ in domain $\mathbf{W}_f$. Draw the chord equal half of the line segment T_5T_7 and parallel to it. The endpoints of the chord are the reflection q-points of the last two **W** prototypes.

Step 6: Fold back all the $\mathbf{W}_i$ prototypes into **W** in the descending order (here, $\mathbf{W}_7 \rightarrow \mathbf{W}_6 \rightarrow \mathbf{W}_5 \rightarrow \mathbf{W}_4 \rightarrow \mathbf{W}_3 \rightarrow \mathbf{W}_2 \rightarrow \mathbf{W}$.

The path is complete. The portions of the virtual path (Figure 8) now form elements of the actual (shortest possible) path in the original limited space (Figure 9).

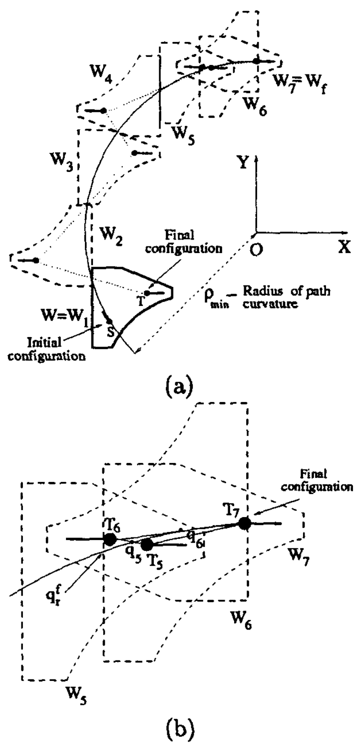

Figure 8: (a) The geometric solution. Six reversals are required to complete the task. (b) Enlarged view of the last two reversals.

9 Discussion

This paper proposes an approach for solving the problem of motion planning for a vehicle operating within a limited two-dimensional workspace, subject to a nonholonomic constraint – the car's path curvature cannot exceed a given value. For given initial and final configurations of the car, the method produces the shortest path possible. The Reflective Unfolding (RU) operator that forms the base of the approach can be effectively used to obtain the optimal path in an analytic form. An important advantage of the method is that it allows one to estimate the path length and its complexity beforehand, based only on the initial and final configuration.

One can also see the presented results as a building block for solving the general problem of motion planning in a constrained environment. Namely, the constraints on the car workspace can in general include obstacles in the car environment. The approach described will fit well those real-world applications in which the area for maneuvering is limited, or where information is obtained via sensors and the environment is known only partially at all times. Then, the vehicle must use the limited (known at the moment) surrounding area to calculate the nearest maneuvers. Or, in the case of obstacles, one can "carve out" a closed domain around the car that is free of obstacles, and calculate the maneuvers necessary to get out of that area.

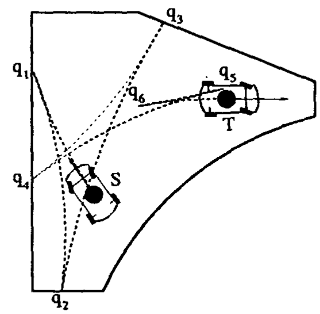

Figure 9: The representation of the shortest path between the initial and final configuration.

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